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k-string tensions and center vortices at large N

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Abstract: We point out that there is a natural explanation, in terms of the center vortex confinement mechanism, for the expected Casimir/Sine Law scaling of k-string tensions in the large-N limit. The crucial ingredient is the existence of Z_N center monopoles, which go over to U(1) monopoles in this limit. Vortex densities leading to Casimir/Sine Law scaling at large-N are constructed; these densities have no obvious pathologies and in particular do not grow with N. We also note that center vortices are stable classical solutions of the Wilson action, for all SU(N) gauge theories with $N > 4$, and extend this old result to a broad class of lattice actions motivated by the improved action program and the renormalization group.

Keywords: Solitons Monopoles and Instantons, 1/N Expansion, Confinement, Lattice Gauge Field Theories.

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Contents

1. Introduction

The term "Casimir scaling" was introduced by the present authors, together with Del Debbio and Faber in ref. [1], as a criterion for the validity of various confinement mechanisms. Casimir scaling refers to the fact, demonstrated numerically in refs. [2, 3], that there is an intermediate distance interval (up to the onset of color screening) where the string tension due to static color sources in color group representation r is approximately proportional to the quadratic Casimir C_r of the representation, i.e.

$$\sigma_r \propto C_r \tag{1.1}$$

This representation dependence is seen quite convincingly in data for the SU(3) group obtained by Bali and by Deldar in refs. [3]. Beyond the Casimir scaling regime, higher color charges are screened by gluons, and the SU(N) asymptotic string tension depends only on the Z_N transformation properties (or "N-ality") of the representation. A correct theory of confinement should be able to explain both Casimir scaling at intermediate distances, where in particular there exists a non-zero adjoint string tension, and the N-ality dependence of the string tension at large distances, where the adjoint string tension vanishes.

For SU(N) gauge theories with $N > 3$ there are a number of color representations, apart from the defining representation and its conjugate, in which color charge cannot be screened by gluons down to charge in a lower dimensional representation. These unscreenable representations each correspond to the lowest dimensional SU(N) representation with N-ality k, represented by a Young tableau with one column of k boxes. Because color screening cannot occur, it is possible that Casimir scaling for these "k-representations" holds asymptotically, and not just in an intermediate range. If so, and if Casimir scaling were exact, the Casimir scaling prediction is that

$$\frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1} \tag{1.2}$$

An alternative scaling law proposed for k-string tensions is the "Sine Law"

$$\frac{\sigma_k}{\sigma_1} = \frac{\sin \pi k/N}{\sin \pi/N} \tag{1.3}$$

which is motivated by MQCD [4].

In fact these two scaling laws are quantitatively not so very different, for $N = 4, 5, 6$. Numerical studies indicate that the Sine Law fits the SU(N) string tension data better than the Casimir Law in $D = 4$ dimensions, while in $D = 3$ dimensions the Casimir Law is the better of the two [5, 6]. Neither formula is a perfect fit in both $D = 3$ and $D = 4$ dimensions. Moreover, as pointed out by Auzzi and Konishi and others [7], the Sine Law cannot be universal among all confining SU(N) theories, since it is found that there are non-universal corrections to this law, for softly broken $\mathcal{N} = 2$ supersymmetric gauge theories.

In the large-N limit, however, Casimir scaling is exact in SU(N) gauge theories due to the well-known factorization property. In this limit there is no difference at all between the Sine and Casimir scaling laws for the k-representations, for k finite as $N \to \infty$. In general, for k-string tensions with $k \ll N$, we have from either law the simple prediction that

$$\frac{\sigma_k}{\sigma_1} = k \qquad (k \ll N) \tag{1.4}$$

We will refer to this limiting behavior of Casimir scaling as "k-scaling". It should be stressed that since k-scaling follows from large-N factorization, this scaling law is guaranteed to hold for large N at intermediate distances (i.e. up to color screening), and perhaps also asymptotically.

The k-scaling law, because it may apply at asymptotic distance scales, poses an interesting challenge to the center vortex theory of confinement. Let us consider SU(N) gauge theory at a very large but finite value of N. The vortex theory leads naturally to the conclusion that the asymptotic string tension depends only on N-ality (i.e. on k, for k-string tensions), and we have suggested in previous work with Faber [8] that Casimir scaling at intermediate distances is due to the finite thickness of vortices. k-scaling, however, may hold at arbitrarily large quark separations, and if so it cannot be attributed to the structure of the vortex core. The interesting question is then whether k-scaling is somehow implicit in the center vortex confinement mechanism.

In this article we would like to offer the following observation: the distribution of center vortices in an SU(N) gauge theory is very likely controlled by an effective Z_N gauge theory, for reasons explained below. This theory is quasi-local at the color-screening scale. As $N \to \infty$, the Z_N theory goes over to a compact U(1) gauge theory, and a Z_N Wilson loop of N-ality k corresponds to the Wilson loop of an object with k units of abelian charge.

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The charge-dependence of string tensions in a U(1) gauge theory, in $D = 3$ dimensions, was worked out some time ago by Ambjørn and Greensite [9]; the same charge dependence is found in Z_N and U(1) lattice gauge theory at strong couplings in any dimension greater than $D = 2$. The dependence is precisely k-scaling, and is due to the formation of k distinct flux tubes forming between the k quark and antiquark. Because of center dominance, the string tension of k-charged loops in the effective U(1) theory is identical to the string tension of k-strings in the corresponding SU(N) theory at large N. In this rather simple way, k-scaling at large N emerges quite naturally from the vortex mechanism, at least in $D = 3$, and probably also in $D = 4$ dimensions.

The following sections will elaborate on this observation, noting in particular the relevance of center monopoles to the dynamics at large N. We will recall and extend a little-known fact, pointed out many years ago by Bachas and Dashen [10], that thin center vortices are stable classical solutions of the Wilson SU(N) lattice action for any $N > 4$. This fact, when extended to effective and renormalization-group improved actions, is potentially of great interest in the context of the vortex confinement mechanism. Finally, we consider vortex densities in the large-N limit. It is shown that vortex densities leading to k-scaling are well-behaved in this limit and, contrary to some recent work [11] based on a dilute gas approximation, do not grow with N.

2. The center monopole gas at large N

Our confidence that the effective vortex theory is a Z_N gauge theory of some kind is based on extensive numerical computations of center-projected observables in SU(2) gauge theory [12]–[15]. Generalizing from our experience with SU(2), we suggest that there is an effective Z_N theory which is obtained from the original SU(N) theory in the following way: begin by fixing the SU(N) lattice theory to an adjoint gauge, where the gauge-fixing condition depends only on the link variables in the adjoint representation. Such adjoint gauges preserve a remnant Z_N symmetry; examples include all the (direct, indirect, ...) variants of maximal center and laplacian center gauges. Each link variable U_μ can be expressed as the product $U_\mu = z_\mu V_\mu$, where $z_\mu \in Z_N$, and $z_\mu I_N$ is the closest center element to U_μ on the group manifold. We can then write

$$\int dU = \sum_{z \in Z_N} \int_{\mathcal{R}} dV \tag{2.1}$$

where the domain $\mathcal{R}$ on the SU(N) manifold consists of group elements V satisfying

$$\mathrm{Re}\,\mathrm{Tr}[V] > \mathrm{Re}\,\mathrm{Tr}[zV] \qquad \text{for all} \quad z \in Z_N \neq 1 \tag{2.2}$$

The effective Z_N theory is then formally defined by integrating over V, i.e.

$$\exp[-S_{\mathrm{eff}}(z_\mu)] \equiv \int_{\mathcal{R}} DV\,\delta\Big[F[U_A]\Big]\Delta[U_A]e^{-S[zV]} \tag{2.3}$$

where the gauge-fixing condition applies to the link configuration U_A in the adjoint representation. It is not necessary to know S_{eff} explicitly in order to simulate the effective theory numerically. Evaluation of a Z_N Wilson loop in the theory defined by $S_{\rm eff}(z)$ is equivalent to the evaluation of a center-projected Wilson loop in SU(N) lattice gauge theory, fixed to an adjoint gauge.

There are some general arguments [16], but no guarantee, that center projection in adjoint gauges will locate confining center vortices. For this property to hold, the center projection must pass (at least) three tests, concerning

1. *Vortex-limited Wilson loops.* Let $W_{(n)}(C)$ denote a Wilson loop in the defining representation of SU(N) gauge theory, evaluated in the subset of configurations satisfying the constraint that in each corresponding center projected configuration, the Z_N loop takes on the value
$$Z(C) = \exp\left[\frac{2i\pi n}{N}\right] \tag{2.4}$$
Then it is required that asymptotically, for large loops,
$$\frac{W_{(n)}(C)}{W_{(0)}(C)} \to \exp\left[\frac{2i\pi n}{N}\right] \tag{2.5}$$
This condition is necessary if a projected loop actually identifies vortices piercing the full loop.

2. *Center Dominance.* The string tension of projected loops of a given N-ality in the adjoint gauge should equal the asymptotic string tension of the corresponding unprojected Wilson loops. This condition tells us that the center vortices located by the projection actually account for the full confining force.

3. *Vortex Removal.* If all center vortices are removed from each full lattice configuration, then the string tension should vanish. This should really be a consequence of the previous conditions, and constitutes a check. In practice, one checks whether the modified loops
$$W'(C) = \langle \mathrm{Tr}[V(C)] \rangle = \langle Z(C)\,\mathrm{Tr}[U(C)] \rangle \tag{2.6}$$
have vanishing asymptotic string tension (where $U(C)$, $V(C)$, $Z(C)$ are products of the full link, V-link, and projected link variables, respectively, around the contour C).

If all three conditions hold, then $S_{\rm eff}(z)$ is the effective action for the confining Z_N flux. The existence of such an effective action is only known to be true from numerical experiments in the case of SU(2) lattice gauge theory, in certain laplacian center gauges.[1] Center dominance appears to work for SU(3) in the original version of laplacian center gauge [18]; the direct and indirect variants have not yet been investigated in the SU(3) case.[2] In the absence of evidence to the contrary, we will assume that an effective Z_N gauge theory satisfying the three conditions above can be defined for *any* SU(N) pure gauge theory via an appropriate choice of adjoint gauge, as in the SU(2) case.

[1]In maximal center gauges, center dominance holds good only to an accuracy of about 20-30% [17]. In variants of laplacian center gauges, the accuracy is better than 10% [15].

[2]There are also some SU(3) results in the direct and indirect maximal center gauge, fixed by over-relaxation [19].

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The effective action $S_{\rm eff}$ is certainly non-local at the lattice scale, but it should be quasi-local at the color-screening scale. This means that the action can be expressed as a sum of loops (and loop products) on the lattice, and the coefficients multiplying large loops in the action must be exponentially suppressed with loop area, for loops whose extension exceeds the color-screening scale. If $S_{\rm eff}$ were not quasi-local, it would be hard to understand how large Z_N Wilson loops could have an area law falloff, or how the theory could avoid long-range correlations.

The excitations of a Z_N gauge theory are thin center vortices and, for $N \geq 3$, center monopoles. A center monopole is the point ($D = 3$) or line ($D = 4$) of intersection of a number of vortices, whose flux adds up to an integer multiple of 2π [20]. For N very large, the Z_N gauge group approximates a U(1) group, and center monopoles go over to the usual abelian monopoles of compact QED. Because the gauge group is compact, Dirac lines/sheets are invisible, and isolated monopoles are stable saddlepoints of the effective action. Since the action is non-local at the lattice scale, the action of the monopole configurations is not set by the lattice cutoff, but rather by the (color-screening) scale at which the effective theory becomes quasi-local.

Let us now consider an isolated monopole solution of the $D = 3$ dimensional U(1) effective theory. The lattice Bianchi identities, coupled with spherical symmetry at scales large compared to the lattice spacing, are sufficient to tell us that the field strength due to a monopole at the origin approximates the continuum form far from the source

$$F^{\rm mon}_{\mu\nu}(x) = \epsilon_{\mu\nu\alpha}\frac{x^\alpha}{|x|^3} + F^{\rm Dirac}_{\mu\nu} \tag{2.7}$$

where $F^{\rm Dirac}_{\mu\nu}$ is the contribution of the Dirac string. A dilute gas of n_m monopoles ($q = 1$) and antimonopoles ($q = -1$) is a superposition of such terms

$$F_{\mu\nu}(x) = \sum_{a=1}^{n_m} q_a \epsilon_{\mu\nu\alpha}\frac{(x-x_a)^\alpha}{|x-x_a|^3} + F^{\rm Dirac}_{\mu\nu}(x) \tag{2.8}$$

The field strength is expressed in terms of variables over a compact range

$$\begin{aligned} F_{\mu\nu}(x) &= A_\nu(x+\widehat{\mu}) - A_\nu(x) - A_\mu(x+\widehat{\nu}) + A_\mu(x) \qquad -\pi \leq A_\alpha \leq \pi \\ &= \widetilde{F}_{\mu\nu} + 2\pi n \qquad \text{where} \quad -\pi \leq \widetilde{F}_{\mu\nu} \leq \pi \end{aligned} \tag{2.9}$$

Since $S_{\rm eff}$ is compact and U(1) invariant, it depends on $\widetilde{F}_{\mu\nu}$ rather than $F_{\mu\nu}$, and Dirac lines contribute nothing to the action. Moreover, if the effective U(1) theory is local at some scale, then a weak-field low-wavelength expansion of the U(1) action in powers of the field strength and its derivatives starts out with the quadratic term, i.e.

$$S_{\rm eff} = \text{const.} + \frac{1}{e^2}\sum_x \widetilde{F}^2_{\mu\nu}(x) \tag{2.10}$$

Following Polyakov [21], and substituting the superposition (2.8) into the weak-field approximation for the effective action, leads to Coulombic interactions between monopoles; i.e. to a monopole Coulomb gas. It should be noted that for a Z_N theory with N finite,

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the center monopole field cannot have the form (2.7) arbitrarily far from the source, since there is a lower limit to the non-zero Z_N flux through a plaquette. In a plasma, however, the field of a monopole is screened at the Debye length, and the monopole Coulomb gas analysis should be valid providing the U(1) approximation (2.7) to the Z_N monopole field holds at least up to the screening length.

It was shown in ref. [9] that a large Wilson loop with k units of abelian charge, in a monopole Coulomb gas, has an area-law falloff with string tension equal to k times the string tension of a single-charged loop. By the center dominance property, the string tension of loops in the effective U(1) theory are the same as the asymptotic string tension of loops of the same U(1) charge (i.e. N-ality for finite N) in the full SU(N) theory. In this way we obtain the k-scaling property in SU(N) in $D = 3$ dimensions.

3. Classical stability of center vortices at $N > 4$

Before turning to the density distribution of vortices in the large-N limit, we would like first to discuss their classical stability in the context of lattice gauge theory.

In the continuum, instantons are stable local minima of the action of pure SU(N) gauge theory. It is a remarkable but little-known fact, pointed out by Bachas and Dashen [10] in 1982, that an analogous result holds in lattice theory: thin center vortices are stable local minima of the Wilson lattice action for any SU(N) gauge group with $N > 4$. The proof is quite trivial: any "thin" center vortex configuration on a classical vacuum background is gauge equivalent, in an SU(N) lattice gauge theory, to a lattice configuration in which each link is a center element, i.e.

$$\begin{aligned} U_\mu(x) &= Z_\mu(x) I_N \\ Z_\mu(x) &= \exp\left[\frac{2\pi i n_\mu(x)}{N}\right] \qquad (n_\mu(x) = 1, 2, \ldots, N-1) \end{aligned} \tag{3.1}$$

It is not hard to see that any vortex-creating singular gauge transformation, operating on the classical vacuum, will result in a configuration which can be transformed into this form. If any plaquette variable is different from unity, then it has been pierced by a vortex. Now consider a small deformation of such thin vortex configurations

$$\begin{aligned} U_\mu(x) &= Z_\mu(x) V_\mu(x) \\ V_\mu(x) &= e^{iA_\mu(x)}, \qquad \mathrm{Tr}[A_\mu^2(x)] \ll 1 \end{aligned} \tag{3.2}$$

The thin vortex is a classical solution if the Wilson action is a (local) minimum for $V_\mu(x)$ gauge equivalent to $V_\mu(x) = I_N$. Substituting (3.2) into the Wilson action for SU(N) gauge theory

$$S = \frac{\beta}{2N} \sum_p \Big(2N - \mathrm{Tr}[U_p] - \mathrm{Tr}[U_p^\dagger]\Big) \tag{3.3}$$

with $\beta = 2N/g^2$, we obtain

$$S = \frac{\beta}{2N} \sum_p \Big(2N - Z_p\,\mathrm{Tr}[V_p] - Z_p^*\,\mathrm{Tr}[V_p^\dagger]\Big) \tag{3.4}$$

Writing the product of V-link variables around a plaquette in the usual way, as the exponential of a field strength

$$V_p = \exp[iF_p] = I_N + iF_p - \frac{1}{2}F_p^2 + \cdots \tag{3.5}$$

and

$$Z_p = e^{2\pi i n_p/N} \tag{3.6}$$

we get, to the leading order in the field strength F_p of the deformation,

$$S = \frac{\beta}{2N}\sum_p \left[2N\left(1-\cos\left(\frac{2\pi n_p}{N}\right)\right) + \cos\left(\frac{2\pi n_p}{N}\right)\mathrm{Tr}[F_p^2] + O(F^3)\right] \tag{3.7}$$

From this expression it is clear that the action is minimized at

$$\mathrm{Tr}[F_p^2] = 0 \tag{3.8}$$

and hence that the vortex configuration (3.1) is stable, *providing* that at each plaquette with $n_p > 0$, the condition

$$\cos\left(\frac{2\pi n_p}{N}\right) > 0 \tag{3.9}$$

is satisfied. Otherwise, the vortex corresponds to a local maximum. The above condition for vortex stability is satisfied iff

$$\frac{n_p}{N} < \frac{1}{4} \qquad \text{or} \qquad \frac{N-n_p}{N} < \frac{1}{4} \tag{3.10}$$

For $N = 2$ and $N = 3$, which are the most studied cases, the stability condition cannot be satisfied, and center vortices are clearly unstable at the classical level. Beginning, however, with $n_p = 1, 4$ at $N = 5$, vortex stability is obtained already from the classical action.

This simple result can of course be extended beyond the simple Wilson action, and therein lies its physical relevance. It is obvious that thin vortices, stable or not, are of no real importance at weak couplings, because they are suppressed by a factor of order

$$\exp\left[-\frac{\text{Vortex Area}}{g^2}\right] \tag{3.11}$$

and do not percolate. The configurations which are of physical interest are center vortices having some finite thickness in physical units. In order to investigate the stability of vortices of thickness d (or of lesser thickness, but including quantum fluctuations up to scale d) starting from a lattice action at spacing a, we imagine following the renormalization group approach, successively applying blocking transformations of the form

$$e^{-S'[\mathcal{U}]} = \int DU\, \delta[\mathcal{U} - F(U)]e^{-S[U]} \tag{3.12}$$

where $\mathcal{U}$ are links on the blocked lattice, and $F(U)$ is a blocking function. The transformations are repeated until a lattice action with lattice spacing d is obtained.

The Monte Carlo Renormalization Group (MCRG), and the closely related perfect action approach, aim to compute effective lattice actions at large scales. It is assumed that only a few contours (plaquettes, 6-link loops, 8-link loops...) are important. Given an effective action at a given length scale d, we can then ask whether there exist stable center vortex solutions. Let us consider, for simplicity, the class of improved lattice actions which consist of plaquette plus 1×2 rectangle terms, i.e.

$$S = c_0 \sum_{\mathrm{plaq}\, P} (N - \mathrm{Re}\,\mathrm{Tr}[U(P)]) + c_1 \sum_{\mathrm{rec}\, R} (N - \mathrm{Re}\,\mathrm{Tr}[U(R)]) \tag{3.13}$$

This class of actions has been widely discussed in the lattice literature, and includes the tadpole-improved action [22], the Iwasaki action [23], and two-parameter approximations [24] to the Symanzik action [25], and the DBW2 action [26]. For $c_0,\ c_1 > 0$, the stability of center vortices is trivial at $N > 4$, and the analysis is exactly like the Wilson action case. However, most improved actions of this type have $c_1 < 0$, and so stability must be reconsidered.

We will now show that the condition for the stability of the trivial vacuum $U_\mu(x) = I_N$ in the two parameter action (3.13), which is

$$c_0 + 8c_1 > 0 \tag{3.14}$$

is also sufficient to guarantee that thin vortices satisfying eq. (3.9) are local minina of the two-parameter action, and therefore classically stable. This stability condition is satisfied by all of the two-parameter improved actions [22]–[26] mentioned above.

It is enough to consider whether the action of a trivial vacuum or vortex configuration can be lowered in a plane by some deformation $V_\mu(x) \neq I_N$ in eq. (3.2). If this cannot be done in a plane, where the Bianchi identity can be ignored and plaquette variables chosen independently to minimize the action, then the action cannot be lowered in a volume either, since the action in a volume is just the action in a set of planes. Then, since we are considering the case that c_1 is negative, we want to restrict our attention to configurations in which the rectangle contributions in eq. (3.13) are as large as they can possibly be, for a specified set of plaquette terms $N - \mathrm{Re}\,\mathrm{Tr}[U_p]$ in the plane. Consider any rectangle R containing plaquettes $p_1,\ p_2$, and write

$$\begin{aligned} U(p_1) &= Z(p_1) \exp[i f_1 \widehat{e}_1 \cdot \vec{L}] \\ &= Z(p_1) \Big(I_N + i f_1 e_1^a L_a - \frac{1}{2} f_1^2 e_1^a e_1^b L_a L_b + \cdots \Big) \\ U(p_2) &= Z(p_2) \exp[i f_2 \widehat{e}_2 \cdot \vec{L}] \\ &= Z(p_2) \Big(I_N + i f_2 e_2^a L_a - \frac{1}{2} f_2^2 e_2^a e_2^b L_a L_b + \cdots \Big) \end{aligned} \tag{3.15}$$

where $\widehat{e}_{1,2}$ are unit vectors, and f_1, f_2 are positive. Then the rectangle term has the form

$$U(R) = U(p_1) g U(p_2) g^\dagger \tag{3.16}$$

where g is an SU(N) group element corresponding to a certain link variable on the rectangle, and in general

$$g U(p_2) g^\dagger = Z(p_2) \Big(I_N + i f_2 e_3^a L_a - \frac{1}{2} f_2^2 e_3^a e_3^b L_a L_b + \cdots \Big) \tag{3.17}$$

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The rectangle variable, to second order in $f_1,\ f_2$, is

$$\begin{aligned} N - \mathrm{Re\,Tr}[U(R)] &= N(1 - \mathrm{Re}[Z(p_1)Z(p_2)]) \\ &\quad + \frac{1}{4}\mathrm{Re}[Z(p_1)Z(p_2)]\Big(f_1^2 + f_2^2 + 2f_1f_2\widehat{e}_1\cdot\widehat{e}_3 + \ldots\Big) \end{aligned} \tag{3.18}$$

and this is clearly as large as possible, for given $f_1, f_2 > 0$ and $\mathrm{Re}[Z(p_1)Z(p_2)] > 0$, when $\widehat{e}_1 = \widehat{e}_3$. The latter condition will always be satisfied if we choose all link variables in the plane to lie in the same U(1) subgroup of SU(N), so it is sufficient, for the purpose of proving stability, to make this choice.

We consider small deformations around a one-vortex configuration, which pierces plane at plaquette p_0, and denote the plaquettes adjacent to p_0, as $\{p_i,\ i = 1-4\}$. Writing again that the deformation around a plaquette is

$$V(p) = \exp[if(p)\widehat{e}\cdot L] \tag{3.19}$$

and defining

$$\begin{aligned} r &\equiv -\frac{c_1}{c_0} \\ z &\equiv \cos\left(\frac{2\pi n_{p_0}}{N}\right) \end{aligned} \tag{3.20}$$

we find the action in the plane S_{plane}, up to second order in the deformation, to be

$$\begin{aligned} Q &\equiv \frac{4}{c_0}\Big(S_{\mathrm{plane}} - N(1-z)(1-4r)\Big) \\ &= \sum_p f^2(p) - r\sum_p\sum_{\mu=1}^{2}(f(p) + f(p+\widehat{\mu}))^2 + \\ &\quad + (1-z)r\sum_{i=1}^{4}(f(p_0) + f(p_i))^2 - (1-z)f^2(p_0) \end{aligned} \tag{3.21}$$

where $p + \widehat{\mu}$ denotes the plaquette adjacent to p in the $\widehat{\mu} = \widehat{1}$ or $\widehat{2}$ directions tangent to the plane. A little rearrangement brings this to the form

$$\begin{aligned} Q &= \sum_p (1-8r)f^2(p) + r\sum_p\sum_\mu (f(p+\widehat{\mu}) - f(p))^2 + \\ &\quad + (1-z)r\sum_{i=1}^{4}(f(p_0) + f(p_i))^2 - (1-z)f^2(p_0) \end{aligned} \tag{3.22}$$

Now if $z = 1$, so the expansion is around the trivial vacuum, then we can immediately read off that Q is minimized at $f(p) = 0$, and the trivial vacuum is stable, providing that $8r < 1$, which is the condition (3.14) above. If, on the other hand, $8r > 1$, then Q has no minimum to second order in the deformation, and the trivial vacuum is unstable.

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Next, separating out of the first two sums in Q the contributions containing the plaquette p_0, we have

$$\begin{aligned} Q &= \sum_{p \neq p_0} (1-8r) f^2(p) + r \sum_{p, p+\widehat{\mu} \neq p_0} (f(p+\widehat{\mu}) - f(p))^2 + \\ &\quad + r \sum_{i=1}^{4} (f(p_i) - f(p_0))^2 + r(1-z) \sum_{i=1}^{4} (f(p_i) + f(p_0))^2 + (z - 8r) f^2(p_0) \\ &= \sum_{p \neq p_0} (1-8r) f^2(p) + r \sum_{p, p+\widehat{\mu} \neq p_0} (f(p+\widehat{\mu}) - f(p))^2 + \\ &\quad + rz \sum_{i=1}^{4} (f(p_i) - f(p_0))^2 + 2r(1-z) \sum_{i=1}^{4} f^2(p_i) + z(1-8r) f^2(p_0) \end{aligned} \tag{3.23}$$

If $z > 0$, which was the stability condition for vortices in the Wilson action, and if $r < 1/8$, which is the stability condition for the trivial vacuum in the two-parameter action, then every term contributing to Q is positive semi-definite. The minimum $Q = 0$ is only obtained at $f(p) = 0$, i.e. at $V(p) = I_N$ everywhere, and the vortex is therefore a stable local minimum of the two-parameter action.

So it appears that the result first obtained by Bachas and Dashen is quite robust: center vortices at $N > 4$ are stable minima of lattice actions in a large region of coupling-constant space associated with improved actions, and it seems unlikely that adding a few more contours, as in certain proposed perfect actions [27, 28], would alter this result.[3] If in fact center vortices remain as local minima of the action all along the renormalization trajectory, then their physical effects must become apparent at some scale. The argument is simply that the Boltzmann suppression factor of a vortex in an effective (or perfect) action at lattice spacing d goes like

$$\exp\left[-\frac{\text{Vortex Area}}{\kappa^2(d)}\right] \tag{3.24}$$

where the vortex area is in lattice units. On the other hand, the entropy factor increases with vortex surface area as

$$\exp\left[+c \times \text{Vortex Area}\right] \tag{3.25}$$

where c is a constant. As d increases, $\kappa^2(d)$ also increases. Eventually entropy wins over action, and vortices at that scale will percolate through the lattice.

So far it would appear that vortices are *only* stable at $N > 4$. At this point we will digress briefly to consider what happens for $N = 2, 3, 4$, following closely the discussion in ref. [29]. The salient point is that the effective long-range action at the color-screening scale should contain contours in the adjoint representation, multiplied by coefficients falling with a perimeter rather than an area law. The reasons for this are easiest to see in the context of the strong-coupling expansion. Let us start with the strong-coupling Wilson action with lattice spacing a, and construct a blocked action with lattice spacing La

$$e^{-S'[\mathcal{U}]} = \int DU \prod_{l'} \delta[\mathcal{U}_{l'} - (UUU \cdots U)_{l'}] e^{-S_W[U]} \tag{3.26}$$

[3] We have not investigated those actions systematically, however.

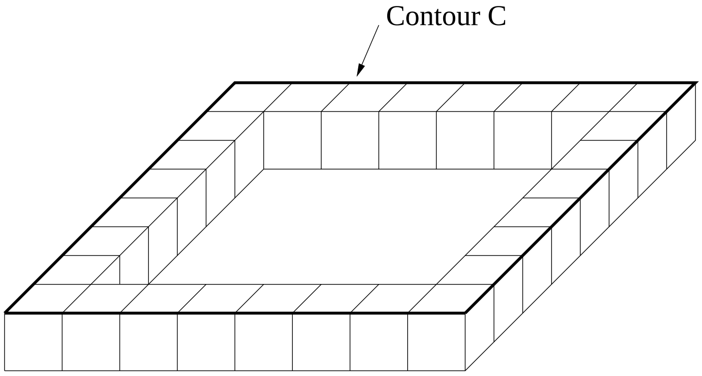

Figure 1: Diagrams leading to perimeter-law falloff, in the blocked action, of adjoint-representation loops.

One readily finds the leading terms in the blocked action

$$-S'[\mathcal{U}] = \sum_{p'} \Big\{ 2N \left(\frac{1}{g^2 N}\right)^{L^2} \mathrm{Re}\,\mathrm{Tr}_F[\mathcal{U}(p')] + \\ + 4(D-2)\left(\frac{1}{g^2 N}\right)^{4(4L-4)} \mathrm{Tr}_A[\mathcal{U}(p')] \Big\} + \\ + \text{larger adjoint loops with } e^{-P} \text{ coefficients} \tag{3.27}$$

The adjoint loops are non-planar contributions, and arise from the "tube" diagrams shown in figure 1. The larger adjoint loops are important for the color-screening of zero N-ality Wilson loops, and cannot be ignored if accurate results for such loops are required.

In fact the non-local blocked action can be expressed (with some additional complications) as a local action containing a number of adjoint Higgs fields [29]. For simplicity, however, we will simply consider the above action truncated to the leading plaquette terms, i.e.

$$-S'_{\mathrm{trunc}}[\mathcal{U}] = \sum_{p'} \Big\{ c_0 \,\mathrm{Re}\,\mathrm{Tr}_F[\mathcal{U}(p')] + c_1 \,\mathrm{Tr}_A[\mathcal{U}(p')] \Big\} \tag{3.28}$$

with

$$c_0 \sim \exp[-\sigma\,\mathrm{Area}(p')]\,, \qquad c_1 \sim \exp[-4\sigma\,\mathrm{Perimeter}(p')] \tag{3.29}$$

Consider small fluctuations around a vortex configuration

$$S'_{\mathrm{trunc}} = \mathrm{const} + \frac{1}{2}\sum_{p'} \left\{ c_0 \cos\left(\frac{2\pi n_{p'}}{N}\right) \mathrm{Tr}_F[F_{p'}^2] + c_1 \,\mathrm{Tr}_A[F_{p'}^2] \right\} \tag{3.30}$$

It is clear that at large blocking L, c_1 is much greater than c_0. Then S'_{trunc}, at the scale where $c_1 \gg c_0$, is minimized at $F_{p'} = 0$. This means that at this scale, vortex configurations are stable even for $N = 2, 3, 4$.[4]

[4] We would like to comment at this point on an argument against vortex stability in Yang-Mills theory (i.e. against any definite thickness for vortices) put forward in ref. [30], and based on an inequality involving

This result was derived for the strong coupling Wilson action. However, from the renormalization-group point of view, the perfect action at the color screening scale (which is approximately one fermi for $N = 2$) is surely a strong-coupling action of some sort, and the same analysis should apply. There is also an interesting study of vortex stability in the framework of continuum QCD, that has been carried out recently by Diakonov and Maul [31].

4. Density of vortices at large N

Numerical simulations in SU(2) lattice gauge theory indicate that a dilute gas approximation for vortices is not really valid in the case of $N = 2$ colors [32], and a "vortex liquid" picture is probably more appropriate. The limitations of the dilute gas approximation, as we will see, are even more apparent at large N.

Following the notation of Del Debbio and Diakonov [11], define an "l-vortex" as a center vortex which, when it pierces the minimal area of a Wilson loop, contributes a factor $z_N^l = \exp[2\pi i l/N]$. A vortex which pierces the loop area twice in opposite directions, and is therefore not topologically linked to the loop, contributes the trivial factor $z_N^l z_N^{N-l} = 1$. The dilute gas approximation assumes that: (i) vortex piercings are statistically independent; (ii) there is no limitation on the number of vortices that can pierce a given area; (iii) it is always possible to distinguish between piercings in a plane due to a single l-vortex, and piercings due to, e.g., l 1-vortices. With these assumptions, the probability that n_l l-vortices pierce a Wilson loop is given by the Poisson distribution

$$P_{n_l} = \frac{\overline{n}_l^{n_l}}{n_l!} e^{-\overline{n}_l} \tag{4.1}$$

where

$$\overline{n}_l = \rho(l, N)\,\mathrm{Area}(C) \tag{4.2}$$

is the average number of l-vortices piercing the minimal area of the Wilson loop, and $\rho(l, N)$ is the number of l-vortices per unit area piercing any given plane. In the dilute gas approximation, the contribution of vortex piercings to an SU(N) Wilson loop in the k-representation is

$$\begin{aligned} W(k, N) &= \prod_{l=1}^{N-1} \sum_{n_l=0}^{\infty} P_{n_l} (z_N)^{k l n_l} \\ &= \exp[-\sigma^{PD}(k)\,\mathrm{Area}(C)] \end{aligned} \tag{4.3}$$

a certain vortex operator. We wish to point out that the argument of that reference could be just as well applied to Z_2 lattice gauge theory at strong couplings, arriving at the same inequality as in the Yang-Mills case, and reaching the false conclusion that Z_2 vortices also have no definite thickness. We believe that the problematic aspect of that analysis was to identify a system in which vortices are restricted to a fixed set of vortex containers, associated with the derived inequality, with the actual vortex vacuum. The restriction of vortices to lie in containers at fixed locations would be, for the vortex vacuum, a drastic truncation of the entropy of vortices due to positional fluctuations. In fact, numerical experiments with both Z_2 lattice gauge theory, and center-projected lattices in Yang-Mills theory, have shown that confinement is due to a single vortex configuration which percolates through the entire lattice. The vortex container of such a configuration has a volume comparable to the volume of the full lattice, but this fact in no way implies that the thickness of the vortex is the length of the lattice.

with the k-string tension

$$\sigma^{PD}(k) = \sum_{l=1}^{N-1} \rho(l,N)\Big(1 - e^{2\pi ikl/N}\Big) \tag{4.4}$$

where $\sigma^{PD}(k)$ and Area(C) are in physical units. The "PD" superscript is a reminder that this string tension is associated with the Poisson distribution.

If one accepts the assumption of a dilute vortex gas, then it is interesting to ask what Casimir or Sine Law scaling implies for the vortex density $\rho(l,N)$. This question was addressed in a recent article by Del Debbio and Diakonov [11], who find that the scaling laws are obtained from eq. (4.4) by the vortex densities (for $l \ll N$)

$$\rho(l,N) = N\sigma(1) \times \begin{cases} \frac{2}{\pi^2(4l^2-1)} + O\left(\frac{1}{N^2}\right) & \text{Sine Law} \\ \frac{N}{2(N-1)\pi^2 l^2} + O\left(\frac{1}{N^2}\right) & \text{Casimir Scaling} \end{cases} \tag{4.5}$$

whose leading terms are proportional to N.

A vortex density growing linearly with N is certainly pathological, but in fact this behavior is not what is found in center vortex theories having the k-scaling property. We will first supply two examples, namely strong-coupling Z_N lattice gauge theory and compact QED_3, which will serve to illustrate this point. Next, we will derive explicit center vortex densities leading to k-scaling, and show that these densities are proportional to $1/N$, rather than N. Finally, the apparent contradiction with the result (4.5) is explained, by showing that finite-range correlations in the center field strength, combined with the k-scaling property, are inconsistent with the assumed Poisson distribution (4.1), from which eq. (4.5) is derived.

We begin with strong-coupling Z_N lattice gauge theory, with Wilson action

$$\begin{aligned} S &= -\beta \sum_p \Big(Z(p) + Z^*(p)\Big) \\ &= -2\beta \sum_p \cos\left[\frac{2\pi n_p}{N}\right] \end{aligned} \tag{4.6}$$

Let $A(C), P(C)$ denote the minimal area and perimeter of loop C in lattice units. For $D = 2$ dimensions, to leading order in β, a Wilson loop in the N-ality k representation (with $k < N/2$) has a vacuum expectation value

$$\langle Z^k(C)\rangle = \left(\frac{\beta^k}{k!}\right)^{A(C)} \qquad (D=2) \tag{4.7}$$

with corresponding string tension (in lattice units)

$$\sigma_k = k\log\left(\frac{1}{\beta}\right) + \log(k!) \tag{4.8}$$

which has k-scaling only for $k \ll \beta$. The result is different for $D > 2$ dimensions. To leading order in β we find instead

$$\langle Z^k(C)\rangle = \beta^{kA(C)+\mu_k P(C)} \tag{4.9}$$

Figure 2: A cross-section of the leading strong-coupling diagram, in $D > 2$ dimensions, for the expectation value of a $k = 5$ Wilson loop in strong-coupling Z_N lattice gauge theory. The loop is perpendicular to the page, crossing it at the open and solid circles. Each solid line is a cross-section of a surface of plaquettes, with surfaces partially overlapping at the right and left-hand boundaries. In hamiltonian formulation, the solid horizontal lines represent k distinct flux tubes running between a k-quark and antiquark.

with a string tension

$$\sigma_k = k\sigma_1 \,, \qquad \sigma_1 = \log\left(\frac{1}{\beta}\right) \qquad (D > 2) \tag{4.10}$$

which has perfect k-scaling for $k < N/2$. The difference between the $D = 2$ and $D > 2$ cases is that for $D = 2$ it is necessary to expand $\exp(\beta Z(p))$ to k-th order in β for every plaquette in the area bounded by loop C. This brings in a factor of $(1/k!)$. By contrast, in any number of dimensions greater than two, it is possible to bring down a single layer of plaquettes on k distinct horizontal surfaces, thereby expanding $\exp(\beta Z(p))$ only to first order at each plaquette on each of these surfaces. A cross-section of the leading strong-coupling diagram, for the case $k = 5$, is shown in figure 2. Overlapping plaquettes at the boundaries account for the perimeter contribution $\exp(-\mu_k P(C))$ in eq. (4.9).

What our simple example demonstrates is that k-scaling readily coexists with center vortex/center monopole confinement mechanisms. Strong-coupling Z_N lattice gauge theory is a theory with only center vortex and (in $D > 2$ dimensions) center monopole excitations; there are no other physical degrees of freedom in the theory. It is also a theory in which the string tension satisfies k-scaling perfectly, for the $D > 2$ case where center monopoles exist.

Let us consider the vortex density $\rho(l, N)$ in strong-coupling Z_N lattice gauge theory, with lattice spacing a. The quantity $\widetilde{\rho}(l, N) = \rho(l, N)a^2$ is the probability that an l-vortex pierces a plaquette. As $\beta \to 0$ all plaquette values are equally likely, i.e. $\widetilde{\rho} \to 1/N$ in this limit, and in general

$$\rho(l, N) = \frac{1}{Na^2}\,[1 + O(\beta)] \tag{4.11}$$

at strong couplings. Far from growing linearly with N, this density actually *falls* with N. Here we have a complete contradiction to the behavior (4.5) derived from the dilute gas approximation. In the special case of $D = 2$ dimensions, where plaquettes all fluctuate independently, the density is obtained immediately from the Wilson action

$$\begin{aligned} \rho(l, N) &= \frac{1}{a^2}\frac{\exp[2\beta\cos(2\pi l/N)]}{\sum_{m=0}^{N-1}\exp[2\beta\cos(2\pi m/N)]} \\ &\approx \frac{1}{Na^2}\frac{\exp[2\beta\cos(2\pi l/N)]}{1 + \beta^2} \end{aligned} \tag{4.12}$$

which illustrates the typical $1/N$ dependence of vortex densities in strong-coupling lattice gauge theory.

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The fact that strong-coupling Z_N lattice gauge theory is a counter-example to the dilute gas result for the vortex density, eq. (4.5), is no doubt due to the fact that the assumptions underlying the dilute gas approximation are not satisfied by the Z_N lattice gauge theory. In particular, in Z_N lattice theory, no more than one vortex can pierce a plaquette. If a Wilson loop around a plaquette has a value $\exp(2\pi il/N)$ with $l \neq 0$, this means that one l-vortex has pierced the plaquette. As a consequence, a planar area $A(C)$ in lattice units can be pierced by no more than $A(C)$ vortices. There is no such restriction in the dilute gas approximation, and the sum over n_l in eq. (4.3) runs to $n_l = \infty$.

A second example of k-scaling by center flux is provided by compact U(1) lattice gauge theory in $D = 3$ dimensions. As explained in section 2, this example is our paradigm for confinement by center degrees of freedom at large N. Once again, this is a theory in which center degrees of freedom are the only degrees of freedom in the theory, since the center of the gauge group is the gauge group itself. In compact QED_3 there is no need to invoke the strong-coupling expansion to demonstrate k-scaling; instead we can apply Polyakov's monopole Coulomb gas analysis to calculate k-string tensions. This analysis was carried out in ref. [9], where it was found that k-scaling is obtained through the existence of k independent surfaces of electric flux (i.e. k separate flux tubes at any given time). As we have seen, this is the same mechanism which accounts for k-scaling in strong-coupling Z_N lattice gauge theory in $D > 2$ dimensions. The formation of k separate flux tubes also accounts for Casimir scaling, in the large N limit, in the gluon-chain model [33].

The example of compact QED_3 also illustrates another important point: the center confinement mechanism does not necessarily depend on having center flux of some fixed magnitude concentrated in tubes (3D) or sheets (4D). Center flux in compact QED_3, passing through a given area in a plane, can spread out through the volume in any manner consistent with the U(1) lattice Bianchi identity; there is no reason for a fixed quantity of center flux to remain concentrated in a tube or a sheet-like region. What *is* essential to the center confinement mechanism is that center flux passing through neighboring regions of a plane, of sufficiently large area, are uncorrelated. In Z_2 lattice gauge theory, where the center flux cannot spread out, the lack of correlation between center flux through neighboring regions is accomplished by percolation of vortices through the entire lattice. In compact QED_3, where the center flux can spread out and the flux through a plaquette can be arbitrarily small, disordering is accomplished via a monopole plasma. In Z_N theories at moderate N, presumably both vortex percolation *and* center monopole effects are in play.

The short-range correlation of center flux is sufficient, in any Z_N or U(1) theory, to derive the area law falloff of Wilson loops. Combined with center dominance, short-range center flux correlation is also is sufficient to derive the area law in SU(N) gauge theories. The derivation for Z_N theories, including the effective P-vortex theory extracted (via eq. (2.3)) from SU(N) gauge theory, goes as follows: we imagine dividing a plane into square regions of area $\mathcal{A}$, with $\mathcal{A}$ taken large enough so that the center flux piercing different regions of the plane are uncorrelated. Then consider a large rectangular loop C in the plane, with minimal area Area$(C) \gg \mathcal{A}$, and subdivide this minimal area into Area$(C)/\mathcal{A}$ adjacent regions of area $\mathcal{A}$ bounded by loops $\{C_i, i = 1, \ldots, \text{Area}(C)/\mathcal{A}\}$. In a

Z_N theory we have the identity

$$Z^k(C) = \prod_i Z^k(C_i) \tag{4.13}$$

Then, using the fact that the flux through the regions are uncorrelated, we have

$$\begin{aligned} \langle Z^k(C)\rangle &= \left\langle \prod_i Z^k(C_i) \right\rangle \\ &= \prod_i \langle Z^k(C_i)\rangle \\ &= \langle Z^k(C_i)\rangle^{\mathrm{Area}(C)/\mathcal{A}} \end{aligned} \tag{4.14}$$

so that

$$\sigma(k) = -\frac{1}{\mathcal{A}} \ln\Big[\langle Z^k(C_i)\rangle\Big] \tag{4.15}$$

where $\sigma(k)$ is the string tension in physical units. Now we define $\widetilde{\rho}(l,\mathcal{A})$ to be the probability that a region of area $\mathcal{A}$ is pierced by flux $2\pi l/N$. We make no distinction about how that flux is divided within the region (e.g. whether the total flux is provided by two vortices of type $l/2$, or one vortex of type l, etc.). Then

$$\langle Z^k(C_i)\rangle = \sum_{l=0}^{N-1} \widetilde{\rho}(l,\mathcal{A}) z_N^{kl} \tag{4.16}$$

and the string tension is

$$\sigma(k) = -\frac{1}{\mathcal{A}} \ln\left[\sum_{l=0}^{N-1} \widetilde{\rho}(l,\mathcal{A}) z_N^{kl}\right] \tag{4.17}$$

If a Z_N lattice gauge theory has both k-scaling and short-range correlations in the flux through a plane – and these conditions hold for both strong-coupling Z_N theory and compact QED_3 – then we can derive the corresponding $\widetilde{\rho}(l,\mathcal{A})$. From eq. (4.16), we see that $\widetilde{\rho}(l,\mathcal{A})$ is the inverse discrete Fourier transform of $\langle Z^k(C_i)\rangle$

$$\widetilde{\rho}(l,\mathcal{A}) = \frac{1}{N} \sum_{k=0}^{N-1} \langle Z^k(C_i)\rangle \; e^{-2\pi ikl/N} \tag{4.18}$$

and inserting the k-scaling behavior

$$\langle Z^k(C)\rangle = \begin{cases} e^{-k\sigma(1)\mathcal{A}} & k \le \frac{N}{2} \\ e^{-(N-k)\sigma(1)\mathcal{A}} & k > \frac{N}{2} \end{cases} \tag{4.19}$$

we find that

$$\begin{aligned} \widetilde{\rho}(l,\mathcal{A}) = \frac{1}{N}\Bigg\{ &\frac{1-\gamma^{[N/2]+1} e^{-2\pi i([N/2]+1)l/N}}{1-\gamma e^{-2\pi il/N}} + \frac{1-\gamma^{[N/2]+1} e^{2\pi i([N/2]+1)l/N}}{1-\gamma e^{2\pi il/N}} - \\ &- 1 - \delta_{2[N/2],N} (-1)^l \gamma^{[N/2]} \Bigg\} \end{aligned}$$

$$= \frac{1}{N}\left\{\frac{(1-\gamma^2)\left(1-(-1)^l\gamma^{[N/2]}\right)}{1+\gamma^2-2\gamma\cos\left(\frac{2\pi l}{N}\right)}\right\} \qquad \text{for even-integer } N$$

$$= \frac{1}{N}\left\{\frac{(1-\gamma)\left(1+\gamma-2(-1)^l\gamma^{[N/2]+1}\cos\left(\frac{\pi l}{N}\right)\right)}{1+\gamma^2-2\gamma\cos\left(\frac{2\pi l}{N}\right)}\right\} \qquad \text{for odd-integer } N \tag{4.20}$$

where we have defined

$$\gamma = e^{-\sigma(1)\mathcal{A}} \qquad \text{and} \qquad \left[\frac{N}{2}\right] \equiv \begin{cases} \frac{N}{2} & \text{even } N \\ \frac{N-1}{2} & \text{odd } N \end{cases} \tag{4.21}$$

It can be checked that for any N, and any $\sigma(1) > 0$,

$$\sum_{l=0}^{N-1} \widetilde{\rho}(l,\mathcal{A}) = 1\,, \qquad \text{and} \quad \widetilde{\rho}(l,\mathcal{A}) > 0 \qquad \text{for every } l \tag{4.22}$$

and one can also verify that in the $\sigma(1) \to 0$ limit, only $\widetilde{\rho}(0,\mathcal{A}) \to 1$ is non-zero (i.e. no string tension means no center flux). Once again we see, in eq. (4.20), the overall factor of $1/N$ multiplying the flux probability distribution, as opposed to an overall factor of N that would have been expected from the dilute gas result (4.5).

Equation (4.20) is a very general result for Z_N gauge theories. It assumes only k-scaling and finite-range flux correlations. The result therefore holds for *any* Z_N theory with these properties, and these include strong-coupling Z_N gauge theory for $D > 2$, compact QED_3 at any coupling, and any other Z_N theory (such as the complicated, non-local P-vortex theory defined in eq. (2.3)) whose long-range structure is described by a monopole Coulomb gas. Our result demonstrates quite clearly that there is no incompatibility whatever between a center flux confinement mechanism and k-scaling of the string tensions.

4.1 Why vortex flux cannot follow a Poisson distribution at large N

The center flux probabilities of eq. (4.20), and the vortex density derived from the Poisson distribution, eq. (4.5), apparently contradict one another. The former expression is proportional to $1/N$ while the latter grows linearly with N, and this discrepancy needs to be explained. We will now show that the combined properties of k-scaling, and the existence of finite-range correlations among center field strengths, are inconsistent with the Poisson distribution (4.1) at large N, from which eq. (4.5) is derived.

The derivation of eq. (4.20) starts from the assumption that one can subdivide the plane into squares of some sufficiently large area $\mathcal{A}$, such that the center fluxes in each area are essentially uncorrelated. In every example we have suggested, where center flux is responsible for confinement, there is a lower bound $\mathcal{A}_{\min} = L_c^2$ to the area $\mathcal{A}$, where L_c is the correlation length for the center field strength. In strong-coupling Z_N lattice gauge theory, $\mathcal{A}_{\min} = a^2$, where a is the lattice spacing, so in this case $L_c = a$. In compact QED_3, L_c is the average monopole separation. In center-projected QCD (the theory defined by $S_{\text{eff}}(z)$ in eq. (2.3)), L_c must be on the order of the characteristic length scale of the theory, i.e. $L_c \sim O(\Lambda_{QCD}^{-1})$. Below these length scales, all field strength fluctuations are correlated, and a treatment of center flux in terms of statistically independent vortex piercings is inconsistent. Thus, to describe the center flux in a plane in terms of statistically independent fluctuations, which is certainly a prerequisite for the use of the Poisson distribution, it is necessary to follow the procedure outlined above: sub-divide the plane into squares of area $\mathcal{A} = \mathcal{A}_{\min}$, and identify the center flux $2\pi l/N$ in a given square with a "piercing" of the square by a vortex of type l.[5] The vortex density is then defined as

$$\rho(l,N) = \frac{\widetilde{\rho}(l,\mathcal{A})}{\mathcal{A}} \tag{4.23}$$

where $\widetilde{\rho}(l,\mathcal{A})$ is the probability that the center flux in the square is $2\pi l/N$. From eq. (4.17), and the fact that the probabilities $\widetilde{\rho}(\lambda,\mathcal{A})$ sum to unity, we have

$$\begin{aligned}
\sigma(k) &= \frac{-1}{\mathcal{A}} \log\left[\widetilde{\rho}(0,\mathcal{A}) + \sum_{l=1}^{N-1} \widetilde{\rho}(l,\mathcal{A}) z_N^{kl}\right] \\
&= \frac{-1}{\mathcal{A}} \log\left[1 - \sum_{l=1}^{N-1} \widetilde{\rho}(l,\mathcal{A}) \left(1 - \exp\left(\frac{2\pi i k l}{N}\right)\right)\right] \\
&= \frac{-1}{\mathcal{A}} \log\left[1 - \mathcal{A}\sigma^{PD}(k)\right]
\end{aligned} \tag{4.24}$$

where $\sigma^{PD}(k)$ is the string tension in eq. (4.4) derived from the Poisson distribution. Equation (4.24) is the correct formula for the string tension $\sigma(k)$, for uncorrelated center flux in areas $\mathcal{A}$. However, the approach based on the Poisson distribution agrees with this correct answer only if all the $\mathcal{A}\sigma^{PD}(k)$ satisfy

$$\mathcal{A}\sigma^{PD}(k) \ll 1 \qquad \text{all} \qquad k \in [0, N-1] \tag{4.25}$$

We can now see that this condition is incompatible with the k-scaling formula

$$\sigma(k) = k\sigma(1) \qquad \left(k < \frac{N}{2}\right) \tag{4.26}$$

at sufficiently large N, and in fact can hold only for

$$N \ll \frac{2}{\mathcal{A}\sigma(1)} \tag{4.27}$$

We conclude that the following conditions

1. k-scaling;
2. finite correlation length $L_c \Rightarrow$ finite lower bound for $\mathcal{A}$;
3. Poisson distribution for vortex piercings;

[5] As explained above in connection with compact QED_3, the term "vortex" in this context should not be taken to mean that the center flux of type l piercing a square region will necessarily remain concentrated in a tube or sheet of constant flux $2\pi l/N$.

are incompatible at sufficiently large N. This means that the result of ref. [11], shown in eq. (4.5) above and indicating that k-scaling requires a vortex density growing linearly with N, is not valid for realistic theories with a finite range of center flux correlations.

An even simpler argument is just to note that $\widetilde{\rho}(0, \mathcal{A}) \sim 1/N$ for fixed $\mathcal{A} = \mathcal{A}_{\min}$ at large N. This means that each square region is virtually certain to be pierced by some finite amount of center flux (as, e.g., in the case of compact QED_3), thereby violating the diluteness assumption underlying the Poisson distribution.

5. Conclusions

The k-scaling of k-string tensions at large N appears to be a natural outcome of a confinement mechanism based on center degrees of freedom. Although k-scaling, as the common large-N limit of Casimir and Sine Law scaling, is certainly not unique to the center vortex scenario, it is still of interest to see how the property emerges in that context. Stated briefly, the confining dynamics at asymptotic distances are described at large N by a center monopole Coulomb gas. The k-string tension, in that system, is proportional to k, because there are k independent flux tubes stretching between the quark and antiquark.

Thin center vortex configurations are stable classical solutions of the SU(N) Wilson lattice action for any $N > 4$, as shown in ref. [10] and reviewed here. We have extended this result to a class of two-parameter lattice actions that are motivated by renormalization-group and tadpole-improvement considerations. It is found that for all couplings such that the trivial vacuum is a stable minimum of the two-parameter action, thin center vortices are also stable local minima of the action. We conjecture that for $N > 4$, thin center vortex configurations are stable minima of the perfect lattice action all along the renormalization trajectory. If that is so, then vortex stability at the semiclassical level is not just a lattice artifact, but is a genuine feature of the continuum theory.

Finally we have shown, by explicit examples, that the k-scaling property does not imply any pathological property of vortex densities in the large-N limit, of the type that was suggested in ref. [11]. Those pathologies derive from the use of a Poisson distribution for vortex densities, which we have shown here to be inconsistent with k-scaling and the existence of a finite correlation length in the large-N limit. We have also derived the (well-behaved) vortex density distribution leading to k-scaling in the large-N limit.

The essential feature of the center confinement mechanism, aside from center dominance, is the finite correlation length of center flux in a plane. Independent fluctuations of the center flux in finite regions guarantees an area law to Wilson loops of non-zero N-ality, with an asymptotic string tension that depends *only* on the N-ality. The necessary randomizing of center flux at finite distances can be achieved via vortex percolation, as in Z_2 or Yang-Mills lattice gauge theory, or from the disordering due to a center monopole plasma, as in Z_N or SU(N) gauge theories at large N. At moderate values of N, perhaps as low as $N = 3$, it is likely that both effects play a role.

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